# The Chain Rule

Math 130 - Essentials of Calculus

15 October 2019

# Differentiating a Composition

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*Using Leibniz notation with $y = f(u)$ and $u = g(x)$, the chain rule becomes*

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

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9. $y(t) = \dfrac{1}{(t^4 + 1)^3}$
10. $k(x) = e^{\sqrt{t^2+2}}$
11. $y = \dfrac{r}{\sqrt{r^2 + 1}}$

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We know that the derivative of $e^x$ is just $e^x$, but what about the derivative of the more generic exponential function: $b^x$, where $b > 0$, $b \neq 1$?

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$$\ln(e^x) = x.$$

Using this, we can rewrite $b = \ln(e^b)$ so that

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Then, using the chain rule, we find that

$$\frac{d}{dx}[b^x] = \frac{d}{dx}[e^{x \ln b}] = e^{x \ln b} \ln b = b^x \ln b.$$

## Example

*Find the derivative of the following functions*

1. $f(x) = 10^x$
2. $g(t) = 2^{3t}$

# The Chain Rule with a Table

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*Here is a table of values for $f$, $g$, $f'$, and $g'$.*

| $x$ | $f(x)$ | $g(x)$ | $f'(x)$ | $g'(x)$ |
|---|---|---|---|---|
| 1 | 3 | 2 | 4 | 6 |
| 2 | 1 | 8 | 5 | 7 |
| 3 | 7 | 2 | 7 | 9 |

*Find the following values*

1. *If $h(x) = f(g(x))$, find $h'(1)$.*

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*Find the following values*

1. If $h(x) = f(g(x))$, find $h'(1)$.
2. If $H(x) = g(f(x))$, find $H'(1)$.
3. If $F(x) = f(f(x))$, find $F'(2)$.
4. If $G(x) = g(g(x))$, find $G'(3)$.